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A short note on the general boundary element method for viscous flows with high Reynolds number

X. Y. Zhao[†] and S. J. Liao[*,‡]

School of Naval Architecture and Ocean Engineering, Shanghai JiaoTong University, Shanghai 200030, China

SUMMARY

In this paper, the general boundary element method and the parallel computation are employed to solve laminar viscous flows in a driven square cavity, governed by the exact Navier–Stokes equations. Using the solution at $Re = 0$ as the initial approximation, the convergent numerical results for high Reynolds number at $Re = 7500$ are obtained, for the first time, by the boundary element method. This verifies the validity and great potential of the general boundary element method for highly non-linear problems, which may greatly enlarge application regions of the boundary element method in science and engineering. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: general BEM; N–S equation; driven cavity flow; parallel computing

1. INTRODUCTION

It is well known that the boundary element method (BEM) often fails to give convergent numerical results of highly non-linear problems. For example, although the problem of the viscous flow in a driven square cavity at high Reynolds number has been solved by iterative numerical techniques [1] of the finite difference method (FDM), the finite volume method (FVM) and so on, to the best of our knowledge, *no* one has successfully employed the BEM to give convergent results at the high Reynolds number $Re = 7500$ (see References [2, 3]). This is mainly because the BEM is based on the superposition of fundamental solutions of linear operators so that it is in principle suitable for linear problems and weakly non-linear problems.

The so-called general boundary element method (GBEM) was proposed by Liao and his co-authors [4–9] to overcome the limitations of the traditional BEM. The GBEM is valid

* Correspondence to: Shijun Liao, School of Naval Architecture and Ocean Engineering, Shanghai JiaoTong University, Shanghai 200030, China.
† E-mail: xyzhao@sjtu.edu.cn
‡ E-mail: sjliao@mail.sjtu.edu.cn

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even if a non-linear equation does not contain any linear terms at all. Thus, the GBEM is more general and is valid for highly non-linear problems. The GBEM is based on an analytic technique for non-linear problems, namely the homotopy analysis method [10–14]. Thus, it has a solid mathematical base. Different from well-known perturbation techniques, the homotopy analysis method is valid even if a non-linear problem does not contain any small parameters at all. Especially, unlike *all* other analytic techniques for non-linear problems, the homotopy analysis method provides us with a simple way to *control* the convergence of approximation series and *adjust* convergence region and rate, as mentioned by Liao [14]. This is the main reason why the GBEM is valid for strong non-linear problems.

It is well known that domain integrals appear when the BEM is employed to solve a non-linear problem, which considerably increase the CPU time. In this paper, we propose a parallel general boundary element method by using parallel computations. To show its validity, our approach is applied to the viscous flow in a driven square cavity, and the convergent numerical results are obtained even at high Reynolds number $Re = 7500$.

2. MATHEMATICAL FORMULATION

Consider laminar viscous flow in a driven cavity, governed by the dimensionless Navier–Stokes equations written in the vorticity ω and streamfunction ψ as follows:

$$\nabla^2\omega = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right) \tag{1}$$

$$\nabla^2\psi + \omega = 0 \tag{2}$$

subject to the boundary conditions

$$\psi = \psi_b \quad \text{on } \Gamma \tag{3}$$

$$\frac{\partial\psi}{\partial n} = \left(\frac{\partial\psi}{\partial n}\right)_b \quad \text{on } \Gamma \tag{4}$$

where Re is the Reynolds number. To the best of our knowledge, no one has successfully applied the boundary element methods to get convergent results of above equations at high Reynolds number $Re = 7500$ (see References [2, 3]).

In 1992 Liao [4] applied the boundary element method to solve the above-mentioned equations and obtained convergent results up to $Re = 2000$ by means of a non-uniform grid 40×40 and the following mth-order iterative formulas:

$$\omega_0^{i+1}(x, y) \leftarrow \omega_0^i(x, y) + \sum_{k=1}^{m} \omega_k^i(x, y)\, p^k \tag{5}$$

$$\psi_0^{i+1}(x, y) \leftarrow \psi_0^i(x, y) + \sum_{k=1}^{m} \psi_k^i(x, y)\, p^k \tag{6}$$

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where $p \in [0, 1]$ is a parameter to be chosen, the superscript i denotes the number of iteration, $\omega_k^i(x, y)$ and $\psi_k^i(x, y)$ are governed by

$$\nabla^2 \omega_k^i(x, y) = (\chi_k - 1)\nabla^2 \omega_{k-1}^i(x, y) + s_{k-1}(x, y) \tag{7}$$

$$\nabla^2 \psi_k^i(x, y) + \omega_k^i(x, y) = (\chi_k - 1)[\nabla^2 \psi_{k-1}^i(x, y) + \omega_{k-1}^i(x, y)] \tag{8}$$

subject to the boundary conditions

$$\psi_k^i(x, y) = 0 \quad \text{on } \Gamma \tag{9}$$

$$\frac{\partial \psi_k^i(x, y)}{\partial n} = 0 \quad \text{on } \Gamma \tag{10}$$

where

$$s_{k-1}(x, y) = Re \sum_{n=0}^{k-1} \left[\frac{\partial \psi_n^i}{\partial y} \frac{\partial \omega_{k-1-n}^i}{\partial x} - \frac{\partial \psi_n^i}{\partial x} \frac{\partial \omega_{k-1-n}^i}{\partial y} \right] \tag{11}$$

and

$$\chi_k = \begin{cases} 0 & \text{when } k \leqslant 1 \\ 1 & \text{when } k > 1 \end{cases} \tag{12}$$

Note that Equations (7) and (8) with boundary conditions (9) and (10) can be easily solved by the traditional BEM. Liao [4] found that, when $m = 1$ in iterative formulas (5) and (6), corresponding to a traditional BEM approach, one *cannot* get convergent results for $Re > 400$, no matter how small the value of p is. However, when $m = 2$, convergent results can be obtained by means of 40×40 grid at Reynolds number up to $Re = 2000$, provided p is properly chosen. For details, please refer to Liao [4].

3. PARALLEL COMPUTING

It is well known that the BEM is efficient for linear differential equations. However, for a nonlinear problem, much more CPU time is needed because the domain integral appears. Recently, parallel computing [15–19] becomes popular for the BEM. Note that it is very efficient and nearly straightforward to apply parallel computations to calculate integrals, mainly because any an integral is in principle parallel and can be divided into many independent integrals on independent sub-domains, i.e.

$$\int\int_S f \, \mathrm{d}s = \int\int_{S_1} f \, \mathrm{d}s + \int\int_{S_2} f \, \mathrm{d}s + \cdots + \int\int_{S_N} f \, \mathrm{d}s$$

where

$$S = S_1 \cup S_2 \cup \cdots \cup S_N, \quad S_1 \cap S_2 \cap \cdots \cap S_N = 0$$

So, it is natural to employ parallel implementations for the GBEM. Here, we focus on parallel computation of the domain integral. Rather than the above-mentioned strategy, the data

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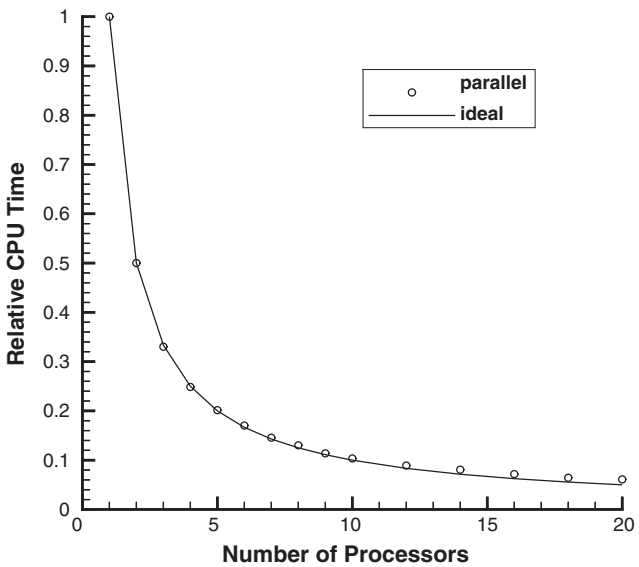

Figure 1. Relative CPU time required for different number of processors on a distributed-memory multicomputer. (mesh: 96×96).

decomposition parallelization strategy [20] is employed, which yields an algorithm with very good load-balancing properties and high efficiency.

The proposed GBEM approach is parallelized by message passing interface (MPI) [21], a widely used standard for writing message passing programs. The choice of message passing library was determined by the requirement that the resulting code is portable to different parallel computing platforms. There are two kinds of parallel computing systems: shared memory and distributed memory ones. The higher efficiency of parallel computation can be obtained by shared memory computing systems, while distributed one is more adaptive.

3.1. Implementation on a distributed memory system

First of all we employ a system of some personal computers (PC) connected by Local Area Net (LAN). Each PC has a processor of Intel Pentium 866 with 128 M memory and is connected by a 10 M hub, while one of them is chosen as the host. In this way we construct an easy-to-use, low-cost computer system available for the parallel computation with data communication between them, called the cluster computing system [22, 23]. Communications by TCP/IP are carried out by the message passing library of MPI.

It is found that high parallel efficiency is obtained even by this low-cost cluster computing system. If the number of processors is not large, the CPU time required is almost inversely proportional to the number of processor, as shown in Figure 1. This is mainly because more than 95% CPU time of our GBEM approach is spent for the integrals. As the number of processors becomes large, the parallel performance decreases, mainly because the time for communications between each PC through LAN increases. The parallel performance has been investigated with two numerical grids, 64×64 and 96×96. It is found that for a given

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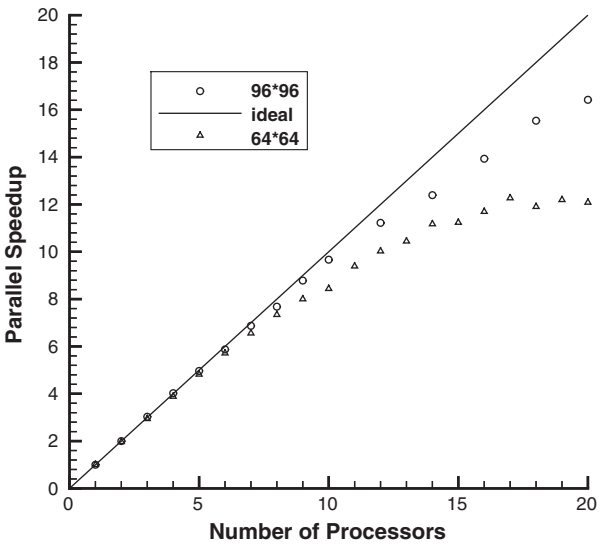

Figure 2. Parallel speed-up for different computational grids and number of processors on a distributed-memory multicomputer.

number of processors, the finer the grid, the more efficient the parallel computation, as shown in Figure 2.

3.2. Implementation on a shared memory system

To test the performance of our parallel GBEM approach on a high-performance parallel computer, we implement our code on SGI Onyx 3800, a supercomputer involving 64 processors at frequency of 500 MHz with 32 GB shared memory. MPI is supported by this machine so that our code can be employed without any changes.

Obviously, the shared memory makes the communications between different processors much faster. Thus, less time for message passing is needed and the efficiency is much higher, as shown in Figure 3. Therefore, the parallel performance of a parallel supercomputer with shared memory is higher than that of the cluster computing system with distributed memory. However, the cost of the latter is much lower than that of the former.

All of our calculations indicate that the parallel computation based on the data decomposition parallelization strategy [20] is indeed rather efficient for the general boundary element method, even by means of a low-cost cluster computing system.

4. NUMERICAL RESULTS

We use a non-uniform rectangle mesh as numerical grid. Since there is discontinuity in the velocity field at corners, a so-called ‘double-node approach’ technique is applied to treat the singularity at corners where there are two nodes rather close to each other but belonging to two different sides. We use linear elements and regular integral on the boundary. The domain

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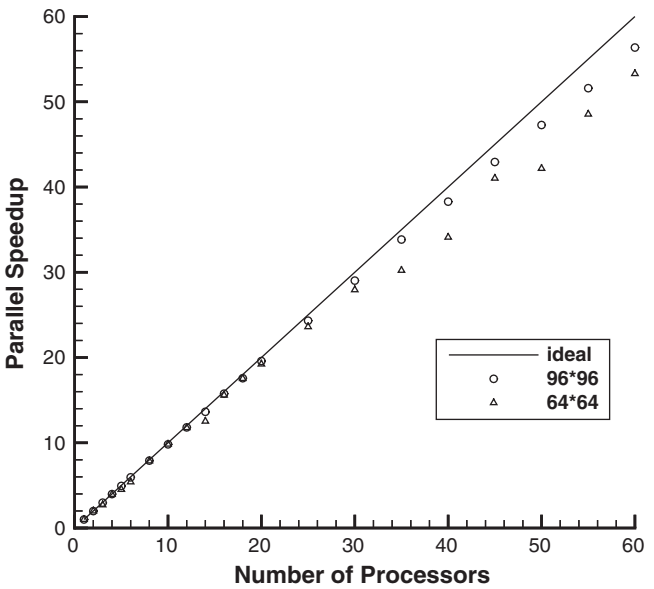

Figure 3. Parallel speed-up for different computational grids and number of processors on SGI Onyx 3800.

Table I. The value of the parameter p.

Re	1000	3200	5000	7500
p	0.05	0.01	0.005	0.002

integrals are calculated by four-point Gauss numerical integral method. For details please refer to Liao [4].

The solution at $Re = 0$ can be obtained without iteration and therefore is used as our initial approximation. All of our numerical solutions for Reynolds number up to $Re = 7500$ are obtained starting from this initial. To ensure the convergence, we follow Liao [4] to use the second-order iterative formulas, namely $m = 2$ in (5) and (6). Note that we have freedom to choose the value of $p \in [0, 1]$, which provides us with a simply way to control the convergence of the iteration. It is found that the value of p had to be decreased as the Reynolds number increases, as shown in Table I. Besides, as the non-linearity becomes stronger, the number of iteration increases, as shown in Figure 4. It is found that, even at high Reynolds number, there always exists a small enough p, by means of which we can gain the correct, convergent results from the initial approximation obtained at $Re = 0$. But, it is unknown right now how to find the best value of p to ensure the best convergence rate. When $m = 1$ in the iterative formulas (5) and (6), corresponding to the traditional BEM, we also fail to get convergent results for $Re > 400$, as reported by Liao [4]. Like Liao [4], we cannot obtain convergent results at $Re = 7500$ by means of the grid 40×40, mainly because a fine enough

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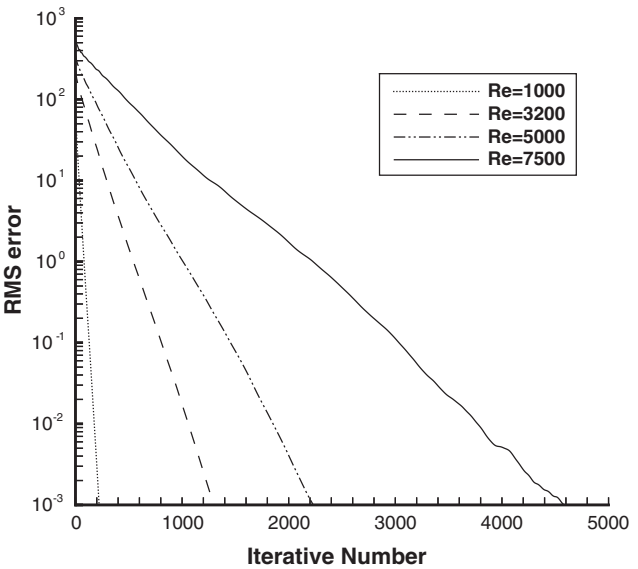

Figure 4. Root-mean-square errors via iterative number (mesh: 64×64).

Table II. Comparison of the results about primary vortex with those given by Ghia *et al.* [24].

	Current results				Results given by Ghia *et al.*			
Re	$\psi_{\max}$	$\omega_{v,c}$	$x_{v,c}$	$y_{v,c}$	$\psi_{\max}$	$\omega_{v,c}$	$x_{v,c}$	$y_{v,c}$
1000	−0.1177	2.0640	0.5333	0.5667	−0.1179	2.0497	0.5313	0.5625
3200	−0.1197	1.9376	0.5177	0.5417	−0.1204	1.9886	0.5165	0.5469
5000	−0.1193	1.9048	0.5177	0.5333	−0.1190	1.8602	0.5117	0.5352
7500	−0.1184	1.8773	0.5083	0.5333	−0.1200	1.8799	0.5117	0.5322

grid had to be used to simulate the complicated flows at high Reynolds number in a driven square cavity.

The numerical results are presented for four different Reynolds numbers: 1000, 3200, 5000 and 7500. The root-mean-square errors via the iterative number are shown in Figure 4. The comparisons of our numerical results given by grid mesh 120×120 with respect to those reported by Ghia *et al.* [24] are shown in Table II. The velocity profiles of u at $x = \frac{1}{2}$ and v at $y = \frac{1}{2}$, compared with the results given by Ghia *et al.* [24], are shown in Figure 5 and 6, respectively. The contours of the stream-function are shown in Figures 7. All of our numerical results agree well with the solutions provided by Ghia *et al.* [24] by mean of the finite difference method.

It should be emphasized that, to the best of our knowledge, *no* one has gained convergent results for the viscous flow in a driven square cavity at $Re = 7500$ by the *boundary element method*. This verifies that the general boundary element method is indeed valid for highly non-linear problems and therefore might greatly enlarge application regions of the boundary element method in science and engineering.

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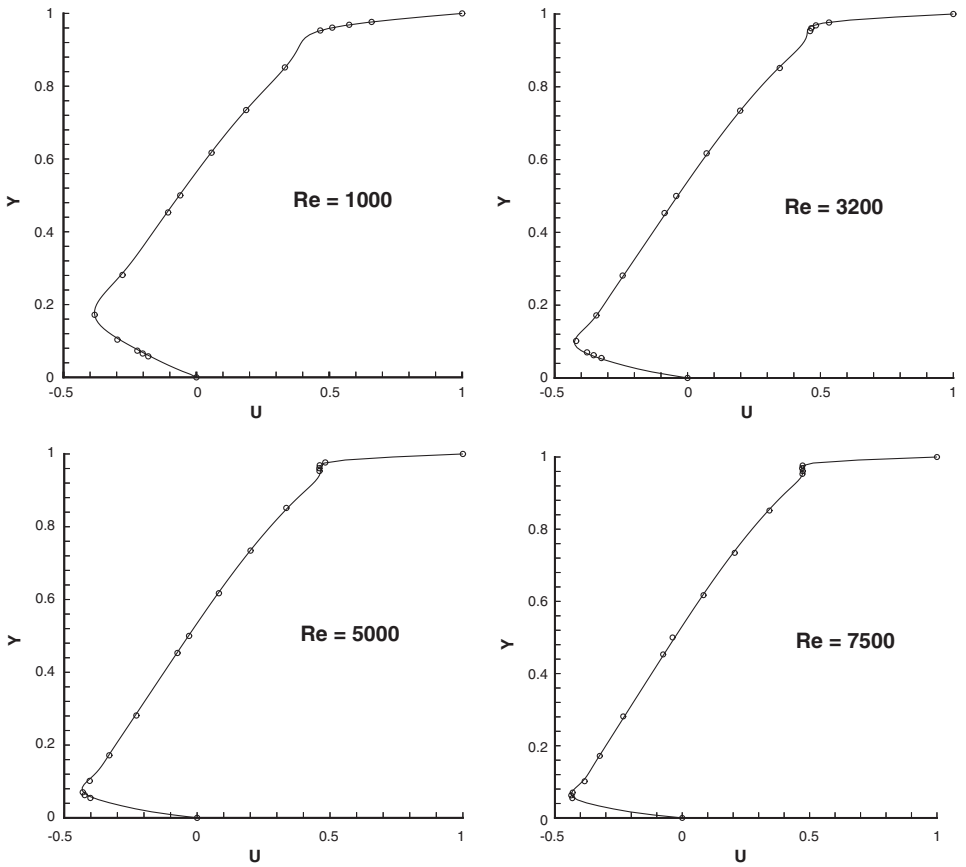

Figure 5. Profiles of velocity u at $x = \frac{1}{2}$ for $Re = 1000, 3200, 5000, 7500$. Solid line: current result; circle: results given by Ghia *et al.* [24].

5. CONCLUSION

In this paper, the general boundary element method and the parallel computation are employed to solve laminar viscous flows in a driven square cavity, governed by the exact Navier–Stokes equations. Using the solution at $Re = 0$ as the initial approximation, the convergent numerical results for high Reynolds number at $Re = 7500$ are obtained, for the first time, by the boundary element method. This verifies the validity and great potential of the general boundary element method for highly non-linear problems, which may greatly enlarge application regions of the boundary element method in science and engineering.

When the boundary element method is employed to non-linear problems, most of CPU time is spent to calculate integrals. Thus, it is certainly very efficient and straightforward to apply the parallel computation to the BEM. On the one side it is easy to make a cluster computing system with low-cost, and on the other side the appearance of MPI makes parallel computation convenient and popular. So, combining the general boundary element method

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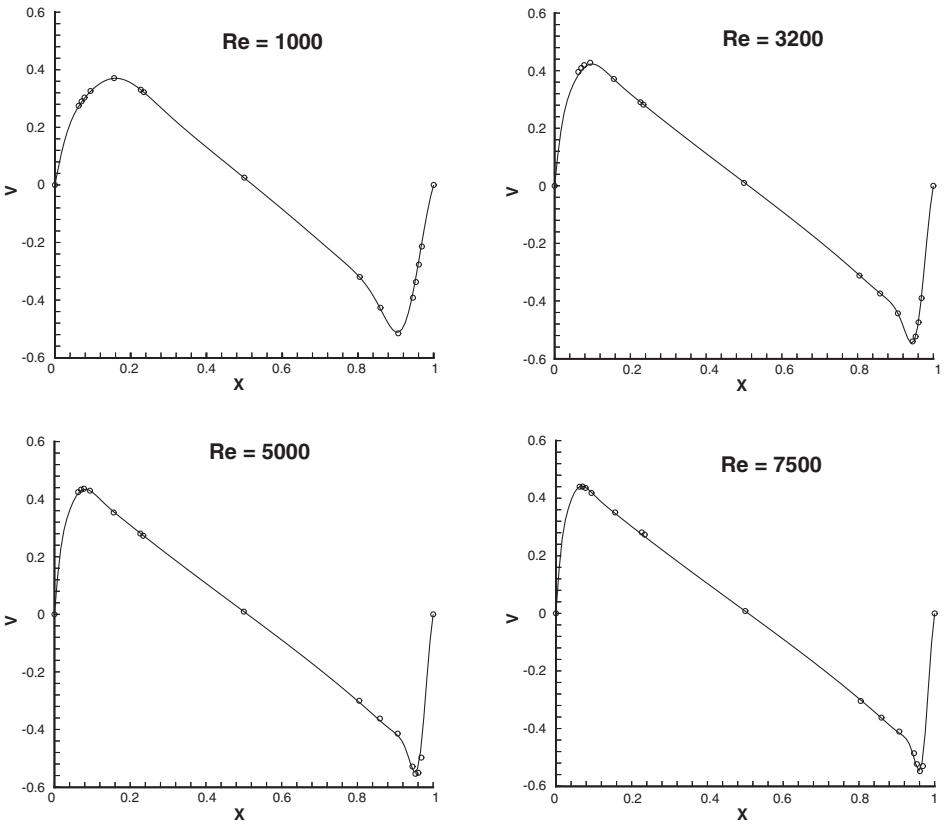

Figure 6. Profiles of velocity v at $y = \frac{1}{2}$ for $Re = 1000, 3200, 5000, 7500$. Solid line: current result; circle: results given by Ghia *et al.* [24].

[4–10] with the above-mentioned techniques for parallel computations, the boundary element method can be more efficiently applied to solve a lot of highly non-linear problems in science and engineering.

Note that *only* the high-order iterative formulas ($m \geqslant 2$) can ensure the convergence of iterations at high Reynolds number $Re = 7500$, provided the value of the parameter p is properly chosen. Although we can always find a small enough value of p to ensure the convergence of iteration at high Reynolds number, it is still an open question how to give the best value of p to gain the best convergence rate *a prior*. Besides, many iterations are needed for the high Reynolds number $Re = 7500$. So, it is worthwhile further increasing the efficiency of the GBEM so that it can be employed to more complicated, three-dimensional non-linear problems.

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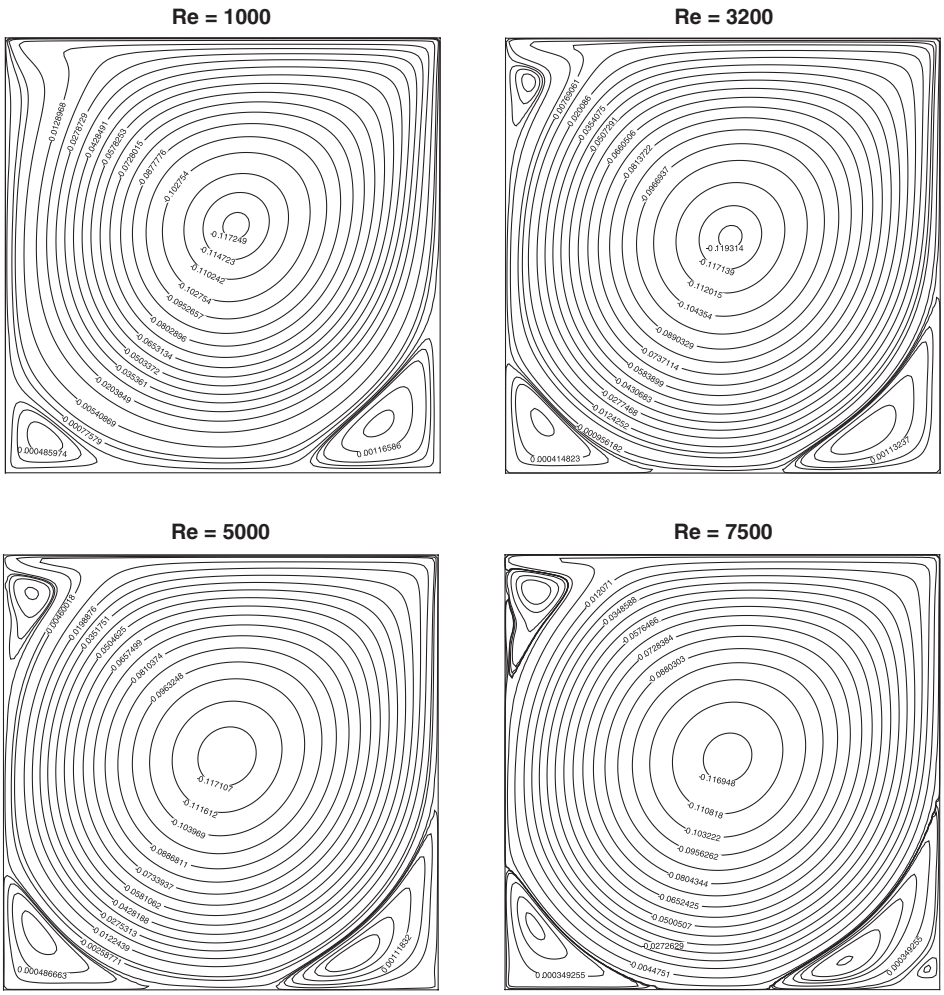

Figure 7. Contour of the stream-function ψ when $Re = 1000, 3200, 5000, 7500$.

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